Notes on Collapses and Persistent Path Homology

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Abstract

In these notes, we examine what is preserved (and what is not) in the transition from the collapse-based simplification of persistent simplicial homology to the persistent path homology of digraphs. We observe, via a counterexample, that an elementary collapse of a path complex does not necessarily preserve path homology. By contrast, the analogue of an elementary *strong* collapse (deletion of a dominated vertex) does preserve path homology.

1 Preliminary concepts

The primary bibliography for this section is [2, 4].

Definition 1.1. A *digraph* is a pair $G = (V, E)$ where $V = \{v_1, \dots, v_n\}$ is a finite set of *vertices* and $E \subseteq V \times V$ is a set of *arcs* (ordered pairs of distinct vertices). For $(v, w) \in E$, we also write $v \to w$.

Definition 1.2. Let V be a finite non-empty set. For any integer $p \geq 0$ an *elementary p-path* on V is a sequence $i_0 \cdots i_p$ of $p + 1$ elements of V (repetitions are allowed); it is *regular* if $i_k \neq i_{k+1}$ for all k. Fixed a field $\mathbb{K}$, we denote by $\Lambda_p = \Lambda_p(V)$ the $\mathbb{K}$-vector space spanned by the regular elementary p-paths, which are now written $e_{i_0 \cdots i_p}$.

Definition 1.3. A *path complex* over V is a set $\mathcal{P}$ of regular elementary paths such that $(v) \in \mathcal{P}$ for every $v \in V$ and such that, whenever $i_0 \cdots i_p \in \mathcal{P}$ with $p \geq 1$, both *truncations* $i_0 \cdots i_{p-1}$ and $i_1 \cdots i_p$ belong to $\mathcal{P}$.

For any allowed elementary path σ and τ, we write $\sigma < \tau$ when σ is a sub-path of τ. Closure under truncation says precisely that P is downward closed for $<$. Also, every digraph G determines a path complex $\mathcal{P}(G)$.

Definition 1.4. For any $p \geq 1$, we define the *p-th boundary operator* $\partial_p : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial_p e_{i_0 \dots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \dots \hat{i_q} \dots i_p}.$$

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Moreover, note that the *interior faces* $i_0 \cdots \widehat{i_q} \cdots i_p$, $0 < q < p$, are in general not elements of $\mathcal{P}$.

Definition 1.5. An elementary p-path $i_0 ... i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, .., p-1$ and *non-allowed* otherwise.

Definition 1.6. Let $\mathcal{A}_p$ the $\mathbb{K}$-linear space spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \mathrm{Span}\{e_{i_0 ... i_p} : i_0 ... i_p \text{ is allowed}\},$$

and consider the following subspace of $\mathcal{A}_p$:

$$\Omega_p := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}. \tag{1}$$

The elements of Ω_p are called *∂-invariant p-paths*. Note that $\Omega_0 = \mathcal{A}_0$, $\Omega_1 = \mathcal{A}_1$, and $\partial \Omega_p \subseteq \Omega_{p-1}$.

Hence, we obtain a chain complex of ∂-invariant paths, $\Omega_\bullet = \Omega_\bullet(G)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \ldots$$

Definition 1.7. The *path homology* of $\mathcal{P}(G)$ is the homology of $(\Omega_\bullet, \partial)$:

$$H_p(G) = H_p(\mathcal{P}(G)) = \frac{\ker \partial\big|_{\Omega_p}}{\mathrm{Im}\, \partial\big|_{\Omega_{p+1}}}.$$

The p-th Betti number is defined as the dimension of the p-th path homology: $\beta_p(G) = \dim H_p(G)$.

A nested family $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_n$ of path complexes gives an inclusion of chain complexes $\Omega_\bullet(\mathcal{P}_i) \hookrightarrow \Omega_\bullet(\mathcal{P}_{i+1})$ and hence a persistence module $\{H_p(\mathcal{P}_i)\}$, the *persistent path homology* (PPH) of the filtration [1]. For a weighted digraph (V, ω), we can consider the filtration $G_\delta = (V, \{(u, v) : \omega(u, v) \geq \delta\})$.

2 Elementary collapses

Let $\mathcal{K}$ be a path complex and let $\sigma < \tau \subset \mathcal{K}$. Consider τ a path of length n ($n+1$ vertices). We say that τ has dimension n and denote $\dim \tau = n$; and since $\sigma < \tau$, $\dim \sigma = n - 1$. Let $\mathcal{L} = \mathcal{K} \backslash \{\sigma, \tau\}$ be the path complex resulting from deleting σ and τ from $\mathcal{K}$. We say that $\mathcal{K}$ collapses onto $\mathcal{L}$ by an *elementary collapse* of dimension n. We denote by $\mathcal{K} \searrow \mathcal{L}$ if $\mathcal{K}$ can be transformed into $\mathcal{L}$ by a finite sequence of elementary collapses. Here σ is required to be a *free face*, i.e., τ is the *only* path of $\mathcal{K}$ properly containing σ; this forces $\dim \tau = \dim \sigma + 1$ and τ maximal. Figure 1 shows an example of an elementary collapse that preserves path homology.

The collapse $\mathcal{K}^\omega \searrow \mathcal{L}^\omega$ *preserves weighted homology* if $H_i(\mathcal{K}^\omega) \cong H_i(\mathcal{L}^\omega)$, $\forall i \geq 0$. In the article [8] was shown that weighted simplicial collapses do not necessarily preserve weighted homology. Nonetheless, Wang et al. [6] showed that persistent homology with field coefficients is independent on weights but it will depend on the weights if the coefficients are in a general ring R (e.g., $\mathbb{Z}$).

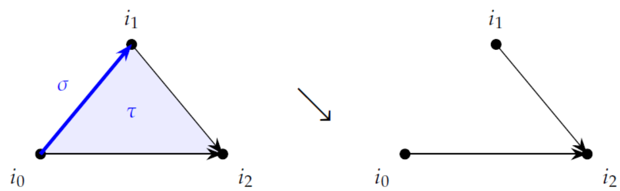

Figure 1: Example of an elementary collapse. In the path complex of the triangle $i_0 \to i_1 \to i_2$, $i_0 \to i_2$, we have $\sigma = i_0 i_1 < \tau = i_0 i_1 i_2$ (τ is maximal). Removing the pair $\{\sigma, \tau\}$ is an elementary collapse, and here it does preserve path homology.

Lemma 2.1. *Let $\mathcal{K} \searrow \mathcal{L}$ be an elementary collapse of path complexes. Then, $\Omega_\bullet(\mathcal{L})$ is a chain subcomplex of $\Omega_\bullet(\mathcal{K})$. Writing $H_*(\mathcal{K}, \mathcal{L}) := H_*\big(\Omega_\bullet(\mathcal{K})/\Omega_\bullet(\mathcal{L})\big)$, there is a long exact sequence $\cdots \to H_p(\mathcal{L}) \to H_p(\mathcal{K}) \to H_p(\mathcal{K}, \mathcal{L}) \to H_{p-1}(\mathcal{L}) \to \cdots$. In particular, the collapse preserves path homology if and only if $H_*(\mathcal{K}, \mathcal{L}) = 0$.*

Proof. By definition, $\mathcal{L} = \mathcal{K} \backslash \{\sigma, \tau\}$, for some $\sigma < \tau$, τ maximal. No $\rho \in \mathcal{L}$ has τ as a truncation (τ is maximal) nor σ (τ is the unique coface of σ), so $\mathcal{L}$ is truncation-closed. For any sub-path-complex $\mathcal{L} \subseteq \mathcal{K}$ one has $\mathcal{A}_p(\mathcal{L}) \subseteq \mathcal{A}_p(\mathcal{K})$, whence $v \in \Omega_p(\mathcal{L})$ gives $\partial v \in \mathcal{A}_{p-1}(\mathcal{L}) \subseteq \mathcal{A}_{p-1}(\mathcal{K})$ and so $v \in \Omega_p(\mathcal{K})$. The sequence is that of the pair. □

The point is that $\Omega_\bullet(\mathcal{K})/\Omega_\bullet(\mathcal{L})$ can be much larger than the two-dimensional quotient produced in the simplicial setting, since deleting σ shrinks $\mathcal{A}_{n-1}$.

In the following, we present a counterexample showing that an elementary collapse may not preserve path homology.

Example 2.2. Let $G = (V, E)$ be a digraph with $V = \{0, 1, 2, 3\}$ and $E = \{0 \to 1,\ 1 \to 2,\ 2 \to 3,\ 0 \to 3, 1 \to 3\}$, where $1 \to 3$ is a chord. The path complex of G is

$$\mathcal{K} = \mathcal{P}(G) = \{0, 1, 2, 3;\ 01, 12, 23, 13, 03;\ 012, 013, 123;\ 0123\}.$$

Consider the paths $\sigma = 13$ and $\tau = 013$. The only path of $\mathcal{K}$ properly containing 13 is 013, and 013 is maximal; so (σ, τ) is a genuine free pair and $\mathcal{L} = \mathcal{K} \setminus \{13, 013\} = \mathcal{P}(G')$, where $G' = G - (1 \to 3)$ (see Figure 2). Then the Betti numbers for the path complexes $\mathcal{K}$ and $\mathcal{L}$ are $\beta_*(\mathcal{K}) = (1, 0, 0)$ and $\beta_*(\mathcal{L}) = (1, 1, 0)$, respectively.

In fact, $\Omega_2(\mathcal{K}) = \langle e_{013}, e_{123} \rangle$ is spanned by the two triangles of G and ∂ carries them isomorphically onto the 2-dimensional cycle space, killing H_1; G is in fact contractible (2 is dominated by 1, leaving the triangle 013). Deleting the arc $1 \to 3$ destroys *both* triangles at once and leaves G' with no double edge, triangle or square, then $\Omega_2(\mathcal{L}) = 0$ by Proposition 2.9 of [3], and G' is the cycle digraph S_4 in its non-contractible orientation, whence $H_1(G') \cong \mathbb{K}$. Thus *an elementary collapse of a path complex can create homology*.

In the previous example, we have $\dim \Omega_\bullet(\mathcal{K}) = (4, 5, 2, 0)$ while $\dim \Omega_\bullet(\mathcal{L}) = (4, 4, 0, 0)$, thus removing a single 2-path costs *two* dimensions of Ω_2.

Definition 2.3. An elementary collapse $\mathcal{K} \searrow \mathcal{L}$ of dimension n is *regular* if $\dim \Omega_p(\mathcal{K}) - \dim \Omega_p(\mathcal{L}) = 1$ for $p \in \{n-1, n\}$ and $= 0$ otherwise.

Proposition 2.4. *If $\mathcal{K} \searrow \mathcal{L}$ is regular and the induced map $\bar{\partial}\colon \Omega_n(\mathcal{K})/\Omega_n(\mathcal{L}) \to \Omega_{n-1}(\mathcal{K})/\Omega_{n-1}(\mathcal{L})$ is nonzero, then $H_*(\mathcal{L}) \cong H_*(\mathcal{K})$.*

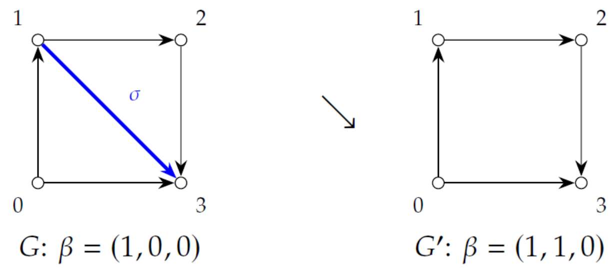

Figure 2: Example 2.2. Collapsing the free pair (13, 013) turns a contractible digraph into the non-contractible 4-cycle.

Proof. Regularity says $\Omega_\bullet(\mathcal{K})/\Omega_\bullet(\mathcal{L})$ is concentrated in degrees $n-1, n$ with both terms of dimension 1; a nonzero map between one-dimensional spaces is an isomorphism, so the quotient is acyclic and Lemma 2.1 applies. □

A degree-based condition of this kind, phrased via discrete gradient vector fields, is obtained in [7], where an $\mathcal{M}$-collapse is shown to preserve path homology when every zero-point of the Morse function has in- and out-degree 1.

3 Strong collapses and persistent path homology

Definition 3.1. A vertex v in $\mathcal{K}$ is called a *dominated vertex* if the link of v in $\mathcal{K}$, $lk_{\mathcal{K}}(v)$, is a simplicial cone, that is, there exists a vertex $v' \neq v$ and a subcomplex $\mathcal{L}$ in $\mathcal{K}$, such that $lk_{\mathcal{K}}(v) = v'\mathcal{L}$. An *elementary strong collapse* is the deletion of a dominated vertex v from $\mathcal{K}$, which we denote by $\mathcal{K} \searrow\searrow^e \mathcal{K}\backslash v$.

Pritam [5] showed that the *persistence homology of a filtration of simplicial complexes is preserved under strong collapse*, i.e., given a filtration of simplicial complexes $\mathcal{K}_i$ connected through inclusion maps

$$Z : \{\mathcal{K}_1 \xrightarrow{f_1} \mathcal{K}_2 \xrightarrow{f_2} \mathcal{K}_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{(n-1)}} \mathcal{K}_n\}, \tag{2}$$

we independently strong collapse the complexes of the filtration to reach a filtration

$$Z^c : \{\mathcal{K}_1^c \xrightarrow{f_1^c} \mathcal{K}_2^c \xrightarrow{f_2^c} \mathcal{K}_3^c \xrightarrow{f_3^c} \cdots \xrightarrow{f_{(n-1)}^c} \mathcal{K}_n^c\}. \tag{3}$$

with the same persistence diagram. The relevant point for us is that strong collapses are governed by *vertex* data, not by the face order, and that is exactly what survives the transition to digraphs.

Definition 3.2. Let G be a digraph and $v \neq v'$ vertices with $v \to v'$ or $v' \to v$. We say v is *dominated by* v' if, for every vertex $u \notin \{v, v'\}$, we have

$$u \to v \implies u \to v' \qquad \text{and} \qquad v \to u \implies v' \to u.$$

In other words, $N^-(v) \setminus \{v'\} \subseteq N^-(v')$ and $N^+(v) \setminus \{v'\} \subseteq N^+(v')$.

Proposition 3.3. *If v is dominated by v' in G, then the vertex map $\rho\colon V \to V \setminus \{v\}$ with $\rho(v) = v'$ and $\rho|_{V\setminus\{v\}} = \mathrm{id}$ is a one-step deformation retraction of G onto the induced subdigraph $G \setminus v$. Consequently $G \simeq G \setminus v$ and $H_*(G) \cong H_*(G \setminus v)$.*

Proof. Definition 3.2 says that ρ sends arcs to arcs or to identified vertices, i.e. that ρ is a digraph map, and it restricts to the identity on $G \setminus v$. Since $\rho(x) = x$ for $x \neq v$ and $v \to v'$ (resp. $v' \to v$), we have $x \to_= \rho(x)$ for all x (resp. $\rho(x) \to_= x$), which is Corollary 3.7 of [3]; homotopy invariance of path homology is their Theorem 3.3. ◻

This is Example 3.15 of [3], where the analogy with simple homotopy theory is already noted. We have restated it to make the contrast with §2 explicit. Removing a dominated vertex always preserves path homology, but removing a free pair does not necessarily do so.

Definition 3.4. Let (V, ω) be a weighted digraph and $G_\delta = (V, \{(u,v) : \omega(u,v) \geq \delta\})$, so that $G_\delta \subseteq G_{\delta'}$ for $\delta \geq \delta'$. We say that v is *uniformly dominated* by v' if v is dominated by v' in G_δ for every δ in the filtration.

Proposition 3.5. *If v is uniformly dominated by v', then the vertex map ρ induces an isomorphism of persistence modules $\{H_p(G_\delta)\} \cong \{H_p(G_\delta \setminus v)\}$ for every p. In particular, the two filtrations have the same persistent path homology barcode.*

Proof. By hypothesis ρ is a digraph map $G_\delta \to G_\delta \setminus v$ at every level, so by Theorem 2.10 of [3], it induces $\rho_{\delta,*}\colon H_p(G_\delta) \to H_p(G_\delta \setminus v)$, an isomorphism by Proposition 3.3. The structure maps of both persistence modules are induced by inclusions, which are the identity on vertices. Since ρ is one and the same vertex map at every level, every square

$$\begin{array}{ccc} H_p(G_\delta) & \longrightarrow & H_p(G_{\delta'}) \\ \downarrow \rho_{\delta,*} & & \downarrow \rho_{\delta',*} \\ H_p(G_\delta \setminus v) & \longrightarrow & H_p(G_{\delta'} \setminus v) \end{array}$$

with $\delta \geq \delta'$, commutes. A level-wise isomorphism commuting with the structure maps is an isomorphism of persistence modules, and isomorphic persistence modules have equal barcodes. ◻

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